CENTRAL WISCONSIN MATHEMATICS LEAGUE

Meet III
March 29, 2001

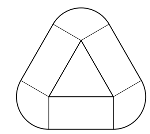
ANSWER KEYS

Category I (Geometry)

1. (a) False (b) True (c) True (d) True (e) False

2. From $\frac{x+5}{8} = \frac{x}{6}$, we get $x = 15$. From $\frac{2y+6}{8} = \frac{x}{6}$, we get $y = \frac{2}{3}x - 3 = 7$.

3. The shaded region is composed of three rectangles and three circular sectors. Since the sectors combine to give a circular disk with radius 1, the area equals $3(1 \cdot 2) + \pi(1)^2 = 6 + \pi$.



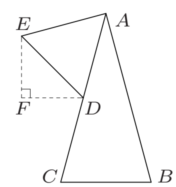
4. If r denotes the radius of the circle, then $11(2r - 11) = (33)(33)$. Solving gives $r = 55$ ft.

5. Note that $\triangle ABD \sim \triangle CAD$. If $x = BD$, then $\frac{x}{6} = \frac{6}{15-x}$ and so $x^2 - 15x + 36 = 0$. Thus $x = 3$ or $x = 12$. Since $\triangle ABD$ is the smaller right triangle in $\triangle ABC$, we want $x = 3$. Therefore $\frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ABC} = \frac{\frac{1}{2}(6)(3)}{\frac{1}{2}(15)(6)} = \frac{1}{5} = 20\%$.

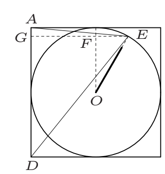
6. The regular hexagon is made up of six congruent equilateral triangles each with height 9 cm. Thus each side has length $\frac{9}{\sqrt{3}/2} = 6\sqrt{3}$ and so the total area is $6\left(\frac{1}{2}(6\sqrt{3})(9)\right) = 162\sqrt{3}\,\text{cm}^2$.

7. $m(\angle ABC) = \frac{1}{2}m(\widehat{AC}) = 40°$. Since $m(\angle AEC) = \frac{1}{2}\left(m(\widehat{AC}) + m(\widehat{BD})\right)$, we get $m(\widehat{BD}) = 110° - 80° = 30°$.

8. Locate F to the left of D and below E so that $m(\angle DFE) = 90°$. Then $m(\angle CDF) = m(\angle ACB) = 75°$. Since $m(\angle ADE) = 60°$, we get $m(\angle EDF) = 45°$. Because $\triangle EDF$ is an isosceles right triangle with a hypotenuse of length 5, it follows that $EF = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$.



9. Let F be at the intersection of 12 o'clock radius and the altitude $\overline{EG}$ of $\triangle AED$. At 1 o'clock, the hour hand is 30° past vertical; thus $\triangle EOF$ is a 30°-60° right triangle. Since $OE = 4$, we get $EG = EF + FG = \frac{1}{2}(4) + 4 = 6$ and so the area of $\triangle AED$ is $\frac{1}{2}(AD)(EG) = \frac{1}{2}(8)(6) = 24\,\text{in}^2$.



10. If $a = BC$, $b = AC$, and $c = AB$, then by using various pairs of similar triangles, it can be shown that $x = \frac{ab}{a+b}$ and $y = \frac{abc}{a^2+ab+b^2}$. Hence $x = \frac{12}{7}$ and $y = \frac{60}{37}$ when $a = 4$, $b = 3$, and $c = 5$.

Category II (Algebra)

1. (a) True (b) True (c) False (d) False (e) False

2. **a** 3. **c** 4. **d** 5. **e**

6. **a** 7. **b** 8. **c** 9. **d**

10. It must be that $x = 3$ is a solution of $x^3 + x^2 + kx - 2k = 0$. Hence $27 + 9 + 3k - 2k = 0$ and so $k = -36$.

11. Complete the square to get $(x-2)^2 + (y+1)^2 = 9$. Therefore the center is $(2,-1)$ and the radius is 3.

12. From $7^{6x-2} = 49^{2x-5} = 7^{2(2x-5)}$ we get $6x-2 = 4x-10$. Solving gives $x = -4$.

13. Let d, c, and p equal the ages (in years) of the duck, the cow, and the pig, respectively. We have $d^2 = c-2$, $c-p = d$, and $d^2 = p$. Solving gives $c = 6$. Therefore the cow is 6 years old.

14. Let d_n denote the rebound height (in meters) after the nth bounce. Then $d_1 = \frac{1}{2}(16), \ldots, d_n = \left(\frac{1}{2}\right)^n(16)$. Therefore $d_8 = \left(\frac{1}{2}\right)^8(16) = \frac{1}{16}$ m.

Category III (Advanced)

1. (a) True (b) True (c) False (d) False (e) True

2. Upon simplifying $\log m + \log n = b$, we get $\log(mn) = b$. This last equation is equivalent to $mn = 10^b$, and so $m = \frac{10^b}{n}$. The answer is **d**.

3. When $h \neq 0$, $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\left(2(x+h)^2 - 3\,(x+h)\right) - \left(2x^2 - 3x\right)}{h} = 4x - 3 + 2h$. The answer is **a**.

4. By following the rules for the order of operations, $\sqrt{2}^{3^{\sqrt{2}^3}}$ is correctly entered into a calculator as $\sqrt{2}\hat{}(3\hat{}(\sqrt{2}\hat{}3))$. This yields about 2321.43. Therefore the nearest integer is 2321.

5. $f(1) = ab = 4$ and $f(2) = ab^2 = 9$. Solving for a and b, we get $a = \frac{16}{9}$ and $b = \frac{9}{4}$.

6. Applying the formula $(a+b)(a-b) = a^2 - b^2$ to $\left((x^2+2^x) + x2^{\frac{x+1}{2}}\right)\left((x^2+2^x) - x2^{\frac{x+1}{2}}\right) - 4^x - 25 = 0$, we get $x^4 - 2x^2 2^x + 4^x - x^2 2^{x+1} - 4^x - 25 = 0$, which simplifies to $x^4 - 25 = 0$. The last equation has $x = \sqrt{5}$ as the only positive real root.

7. Rewrite the equation $\sin^{2000} x - \cos^{2000} x = 1$ as $\sin^{2000} x = 1 + \cos^{2000} x$. Since $\sin^{2000} x \leqslant 1$ and $\cos^{2000} x \geqslant 0$, we conclude that $\cos x = 0$. Therefore $x = \frac{\pi}{2}$ is the only solution in the interval $[0, \pi]$.

8. Since each player has an equal chance to win the game and there are five players, the probability that Player 1 will win the first game is $\frac{1}{5}$.

9. The numbers x_1 and x_2 must be the roots of the equation $x^2 - x - 6 = 6$. Since $x^2 - x - 6 = 6$ is equivalent to $(x-4)(x+3) = 0$, we get $x_1 = -3$ and $x_2 = 4$. Therefore the horizontal distance between $P(-3,6)$ and $Q(4,6)$ equals $|x_1 - x_2| = |-3-4| = 7$.

10. By observing that the period equals 12, we may model the number of daylight hours by the function $f(t) = a\cos\left(\frac{2\pi}{12}t\right) + b = a\cos\left(\frac{\pi}{6}t\right) + b$. Using the fact that $f(0) = a + b = 14$ and $f(6) = -a + b = \frac{28}{3}$, we find that $a = \frac{7}{3}$ and $b = \frac{35}{3}$. The daylight one month after midsummer is therefore $f(1) = \frac{7}{3}\cos\left(\frac{\pi}{6}\cdot 1\right) + \frac{35}{3} \approx 13.6874$ hours $\approx$ 13 hours and 41 minutes.

11. The minimum height is $8\sin 20^\circ$ and the maximum height is $8\sin 50^\circ$, so the difference is $8\sin 50^\circ - 8\sin 20^\circ \approx$ 3.4 m.