CENTRAL WISCONSIN MATHEMATICS LEAGUE
ANSWER KEYS

Meet I — November 20, 2019

Category I (Geometry, No Calculators)

1. (a) **False** (b) **True** (c) **False** (d) **False** (e) **True**

2. If C is the circle's circumference in cm, then
$$\frac{20^\circ}{360^\circ} = \frac{10 \text{ cm}}{C}.$$
Solve the equation to get $C = 180$ cm.

3. If x is the degree measure of $\angle AMN$, then $\angle NMB$ measures $180^\circ - x$. Since $m(\angle OMN) = x/2$, $m(\angle NMP) = (180^\circ - x)/2$, and $m(\angle OMP) = m(\angle OMN) + m(\angle NMP)$, the answer is $(x + 180^\circ - x)/2 = 90^\circ$.

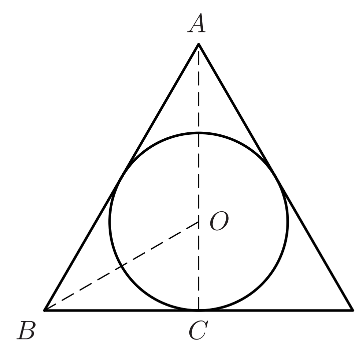
4. Two solutions are given.

   [A] Let x and y denote the width and length of the rectangle in meters with $x \le y$. Solve the system of equations
$$\begin{cases} x + y = \dfrac{42}{2} = 21, \\ xy = 108 \end{cases}$$
to get $x = 9$ and $y = 12$. Therefore, the length of a diagonal is $\sqrt{x^2 + y^2} = \sqrt{9^2 + 12^2} = \sqrt{225} = 15$ meters.

   [B] If x and y denote the width and length of the rectangle in meters, then $x + y = 42/2 = 21$ and $xy = 108$. Therefore, the length of a diagonal is
$$\sqrt{x^2 + y^2} = \sqrt{(x + y)^2 - 2xy} = \sqrt{21^2 - 2(108)} = \sqrt{3^2 \cdot 49 - 3^2 \cdot 24} = 3\sqrt{25} = 15 \text{ meters.}$$

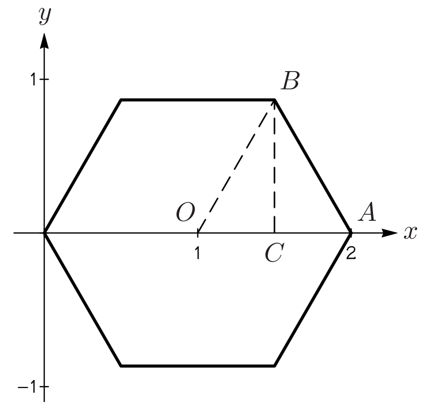
5. Solve $7x - 10^\circ = 180^\circ - 50^\circ$ to get $x = 20^\circ$.

6. Let O denote the center of the circle and let C denote the midpoint of the triangle's base (see the figure to the right). Since $OC = 1$ unit and $\triangle BOC$ is a 30°-60°-90° triangle, it follows that $BC = \sqrt{3}$ units and $OB = 2$ units. By symmetry, observe that $OA = OB$. Because $AC = OC + OA$, we can conclude that the equilateral triangle has area $(AC)(BC) = (OC + OA)(BC) = (1 + 2)\sqrt{3} = 3\sqrt{3}$ square units.

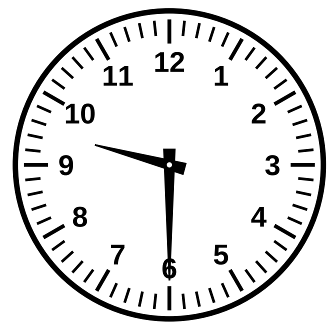


7. If a regular polygon has n sides, then each exterior angle measures $360^\circ/n$ based on how much you rotate at each vertex when making one complete traversal of the regular n-gon. Being supplementary, each interior angle then measures $180^\circ - 360^\circ/n$ and therefore a regular dodecagon ($n = 12$) has interior angles that each measure $180^\circ - 360^\circ/12 = 180^\circ - 30^\circ = 150^\circ$, a regular nonagon ($n = 9$) has interior angles that each measure $180^\circ - 360^\circ/9 = 180^\circ - 40^\circ = 140^\circ$, and a regular octagon ($n = 8$) has interior angles that each measure $180^\circ - 360^\circ/8 = 180^\circ - 45^\circ = 135^\circ$.

8. Let O denote the hexagon's center at $(1,0)$. Then the vertex opposite A must be located at $(0,0)$. Now let C be the midpoint of $\overline{OA}$ (see the figure to the right). Then C is located at $(3/2,0)$. If B denotes the vertex in the first quadrant next to A as shown, then $\triangle BOC$ is a $30°$-$60°$-$90°$ triangle whose hypotenuse $\overline{OB}$ has length 1. Then $OC = 1/2$ and $CB = \sqrt{3}/2$, and so B is located at $(3/2, \sqrt{3}/2)$. The remaining three vertices can be located by symmetry. The desired five vertices are located at $(3/2, \sqrt{3}/2)$, $(1/2, \sqrt{3}/2)$, $(0,0)$, $(1/2, -\sqrt{3}/2)$, and $(3/2, -\sqrt{3}/2)$.



9. The hour hand is halfway between the 9 and the 10 on the clock's face, and the minute hand points toward the 6 (see the figure to the right). Since the 9 and 10 are separated by a central angle that measures $360°/12 = 30°$, and the 9 and 6 are separated by $90°$, the answer is $15° + 90° = 105°$.



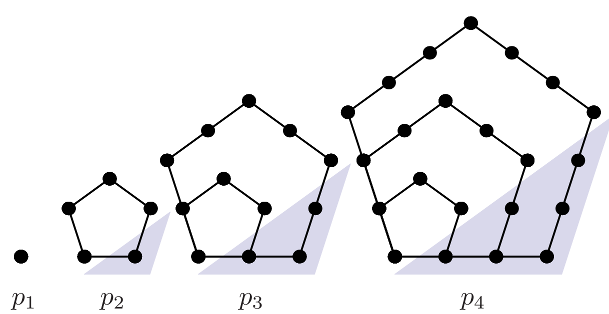
10. Three solutions are given.

[A] Let p_n denote the nth pentagonal number for $n \geq 1$. Observe that the successive differences $p_{n+1} - p_n$ start at 4 and increase by 3 thereafter. Therefore,

$$\begin{aligned}
p_1 &= 1, \\
p_2 &= 1 + 4 = 5, \\
p_3 &= 5 + 7 = 12, \\
p_4 &= 12 + 10 = 22, \\
p_5 &= 22 + 13 = 35, \\
p_6 &= 35 + 16 = 51, \\
p_7 &= 51 + 19 = 70, \\
p_8 &= 70 + 22 = 92, \\
p_9 &= 92 + 25 = 117, \\
p_{10} &= 117 + 28 = 145,
\end{aligned}$$

and so the 10^{th} pentagonal number is 145.

[B] Let p_n denote the nth pentagonal number for $n \geq 1$. As illustrated below, the points in the figure for p_n can be divided into two distinct groups: those outside the shaded triangle and those inside.

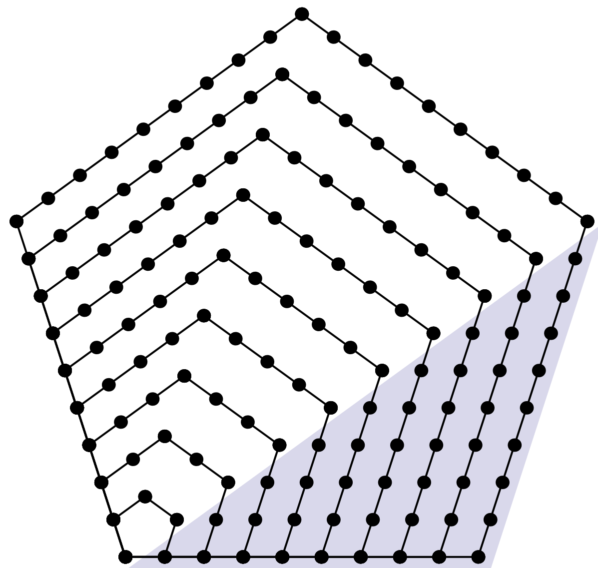


If s_n is the number of points outside the shaded triangle in the figure for p_n, then $s_n = n^2$ (such numbers are called *square numbers*). If t_{n-1} is the number of points inside the shaded triangle in the figure for p_n, then $t_{n-1} = 0 + \cdots + (n-1) \overset{\text{or}}{=} n(n-1)/2$ (such numbers are called *triangular numbers*). Then

$$p_n = s_n + t_{n-1} = n^2 + \big(0 + \cdots + (n-1)\big) \overset{\text{or}}{=} n^2 + \frac{n(n-1)}{2} = \frac{2n^2 + n^2 - n}{2} = \frac{n(3n-1)}{2}$$

and therefore the 10^{th} pentagonal number is $p_{10} = 10^2 + (1 + 2 + 3 + \cdots + 9) = 145$.

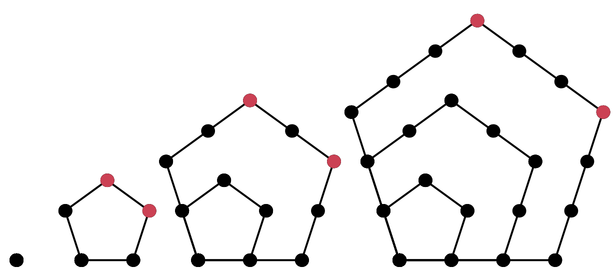
Remark. The figure below is for p_{10}.



p_{10}

$\triangleleft$

[C] Let p_n denote the nth pentagonal number for $n \geq 1$. As illustrated below, the figure for p_n is evidentally contained in the figure for p_{n+1}.



p_1 p_2 p_3 p_4

The points in the figure for p_{n+1} that are not found in the figure for p_n lie on three sides of the largest pentagon that each contain $n+1$ extra points. Being careful not to count extra points more than once (see the two red points in the figures for p_2, p_3, and p_4), the number of extra points is $3(n+1) - 2$ and hence $p_{n+1} = p_n + 3(n+1) - 2 = p_n + 3n + 1$. Since $p_1 = 1$ and $p_{n+1} = p_n + 3n + 1$ for $n \geq 1$, we obtain

$$\begin{aligned}
p_1 &= 1, \\
p_2 &= 1 + \big(3(1) + 1\big) = 5, \\
p_3 &= 5 + \big(3(2) + 1\big) = 12, \\
p_4 &= 12 + \big(3(3) + 1\big) = 22, \\
p_5 &= 22 + \big(3(4) + 1\big) = 35, \\
p_6 &= 35 + \big(3(5) + 1\big) = 51, \\
p_7 &= 51 + \big(3(6) + 1\big) = 70, \\
p_8 &= 70 + \big(3(7) + 1\big) = 92, \\
p_9 &= 92 + \big(3(8) + 1\big) = 117, \\
p_{10} &= 117 + \big(3(9) + 1\big) = 145,
\end{aligned}$$

and so the 10^{th} pentagonal number is 145.

11. Point T is initially translated to the point at $(-2+7, -5) = (5, -5)$. Then the point at $(5, -5)$ is translated to the point at $(5, -5+8) = (5, 3)$. Finally, the point at $(5, 3)$ is reflected about the x-axis to the point at $(5, -3)$.

Category II (Algebra, No Calculators)

1. **d** 2. **b** 3. **d** 4. **c** 5. **a** 6.[1] **d**

7. If n is the lesser integer, then $2019 = n + (n+1) = 2n + 1$. Therefore, $n = 2018/2 = 1009$.

8. If x is the first number and s is the sum of the other two numbers, then $x + s = 275$. Because $x = (150\%)s = 3s/2$, we get $275 - x = s = 2x/3$ and therefore $x = 3(275)/5 = 3(55) = 165$.

9. Recall that $\sqrt{x}$ increases as x increases. Based on the observation that $40^2 = 1600 < 2019 < 2500 = 50^2$, you might guess that $\sqrt{2019} \approx (40+50)/2 = 45$. Observe that $45^2 = 2025 > 2019$ implies that $45 > \sqrt{2019}$. Furthermore, $44^2 = 1936 < 2019$ implies that $44 < \sqrt{2019}$. The answer is 45 since $44.5^2 = 1980.25 < 2019$.

10. Paul rode $1 + 3 + 5 + \cdots + 59 = 900$ miles. Besides manually adding up all the miles ridden, observe that the sum must equal $30(1+59)/2 = 30^2 = 900$ since the daily miles ridden are the first thirty terms of an arithmetic sequence whose first term is 1 and common difference is 2.

    Remark. If n is a positive integer, then $1 + 3 + 5 + \cdots + (2n-1) = n^2$. One way to see why this is so is to let s be the sum of the n terms and then add the terms in the reverse order. Specifically,

$$\begin{array}{rcccccccc}
s = & 1 & + & 3 & + \cdots + & (2n-3) & + & (2n-1) \\
s = & (2n-1) & + & (2n-3) & + \cdots + & 3 & + & 1 \\
\hline
2s = & 2n & + & 2n & + \cdots + & 2n & + & 2n
\end{array}$$

    allows us to conclude that $s = n(2n)/2 = n^2$. More generally, this technique is commonly used to establish that the sum s_n of the first n terms of an arithmetic sequence whose first term is a and common difference is d (so that $a_k = a + (k-1)d$ is the kth term for $k \geq 1$) is given by

$$s_n = n\left(\frac{2a + (n-1)d}{2}\right) = n\left(\frac{a_1 + a_n}{2}\right).$$

◁

[1] View the video at https://www.numberphile.com/videos/the-mystery-of-42-is-solved. Additional information can be found at https://www.bristol.ac.uk/news/2019/september/sum-of-three-cubes-.html.

11. Since $x \neq 3$ we get

$$\frac{x^3 - 3x^2 - 2x + 6}{x - 3} = \frac{x^2(x-3) - 2(x-3)}{x-3} = \frac{(x^2-2)(x-3)}{x-3} = x^2 - 2 = 2019 - 2 = 2017.$$

12. There are five written digits that are even when the odd integers are in the 20's (see the fragment 2123252729 of digits). Similarly, there are five even digits when the odd integers are in the 40's, 60's, 80's, 100's, 120's, 140's, 160's, and 180's. Therefore, there are a total of $5(9) = 45$ written digits that are even.

13. There are twenty-one two-digit prime numbers: 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97. Of those prime numbers, there are nine for which the numbers formed by interchanging the digits are prime: 11, 13, 17, 31, 37, 71, 73, 79, and 97. Therefore, the answer is 9.

14. The given equation can be written as $a^b = 1$, where $a = (x-3)^2 \geq 0$ and $b = x - 3$. Recall that if $a > 0$, then $a^b = 1$ implies that $a = 1$ or $b = 0$. For this problem, $a = (x-3)^2 = 1$ whenever $x = 2$ or $x = 4$, while $b = x - 3 = 0$ or $a = (x-3)^2 = 0$ will not result in a solution for x since 0^0 is undefined. Therefore, the sum of the real solutions is $2 + 4 = 6$.

Category III (Advanced Mathematics, No Calculators)

1. (a) **True** (b) **True** (c) **False** (d) **False** (e) **False**

2. Two solutions are given.

   [A] Recall that the x-coordinate of the vertex of the parabola given by $y = ax^2 + bx + c$ is $x = -b/(2a)$. In the present situation, $x = -6/2 = -3$ and $y = (-3)^2 + 6(-3) + 5 = -4$. The vertex is $(x, y) = (-3, -4)$.

   [B] Complete the square to get $x^2 + 6x + 5 = (x+3)^2 - 4$. Therefore, the vertex is $(x, y) = (-3, -4)$.

3. Two solutions are given.

   [A] Multiply both the numerator and denominator by $\sqrt{3} - \sqrt{2}$ to get

$$\frac{\left(5 + 2\sqrt{6}\right)\left(\sqrt{3} - \sqrt{2}\right)}{\sqrt{2} + \sqrt{3}} = \frac{\left(5 + 2\sqrt{6}\right)\left(\sqrt{3} - \sqrt{2}\right)^2}{\left(\sqrt{2} + \sqrt{3}\right)\left(\sqrt{3} - \sqrt{2}\right)} = \frac{\left(5 + 2\sqrt{6}\right)\left(5 - 2\sqrt{6}\right)}{1} = 1.$$

   [B] Multiply out the numerator to get

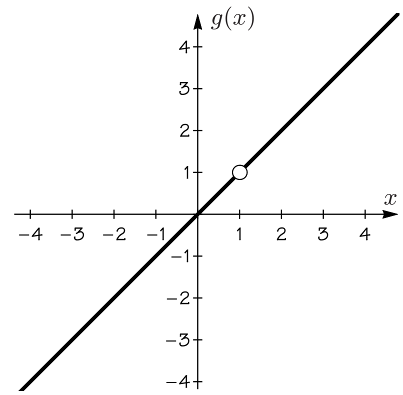
$$\frac{\left(5 + 2\sqrt{6}\right)\left(\sqrt{3} - \sqrt{2}\right)}{\sqrt{2} + \sqrt{3}} = \frac{5\sqrt{3} - 5\sqrt{2} + 2\sqrt{18} - 2\sqrt{12}}{\sqrt{2} + \sqrt{3}} = \frac{5\sqrt{3} - 5\sqrt{2} + 6\sqrt{2} - 4\sqrt{3}}{\sqrt{2} + \sqrt{3}} = \frac{\sqrt{2} + \sqrt{3}}{\sqrt{2} + \sqrt{3}} = 1.$$

4. $\dfrac{6^n + 3^n + 2^{n+1} + 2}{2^n + 1} = \dfrac{\left(2^n + 1\right)\left(3^n + 2\right)}{2^n + 1} = 3^n + 2$

5. Observe that $g(1)$ is undefined and

$$g(x) = \frac{(x-1)\big[(x-1)+1\big]}{(x-1)} = x$$

when $x \neq 1$. Therefore, the graph of g is the line with slope 1 through the origin that consists of all points (x, x) except the point $(1, 1)$, where there is a hole. See the figure to the right.



6. We have $2f(2x) + 2 = 2\left(\dfrac{5}{2x-1}\right) + 2 = \dfrac{10}{2x-1} + 2 = \dfrac{10 + 2(2x-1)}{2x-1} = \dfrac{4x+8}{2x-1} = \dfrac{4(x+2)}{2x-1}$. Acceptable answers are either $\dfrac{10}{2x-1} + 2$, $\dfrac{4x+8}{2x-1}$, or $\dfrac{4(x+2)}{2x-1}$.

7. Let $t = \sqrt[4]{x + 2019}$. Then $t \geq 0$ and $t^2 + t - 6 = 0$. By factoring, we get $(t+3)(t-2) = 0$ and so it must be that $t = 2$. Hence $x + 2019 = 2^4 = 16$ and therefore $x = 16 - 2019 = -2003$.

8. Three solutions are given.

[A] If n is a positive integer, the ones digit of 3^n is
- 1 if n is divisible by 4,
- 3 if the remainder is 1 when n is divided by 4,
- 9 if the remainder is 2 when n is divided by 4, and
- 7 if the remainder is 3 when n is divided by 4.

Since 2019 has a remainder of 3 when divided by 4, the ones digit for 3^{2019} is 7 and therefore the remainder when 3^{2019} is divided by 5 is 2.

[B] Observe that
- the remainder is 3 when 3^1 is divided by 5,
- the remainder is 4 when 3^2 is divided by 5,
- the remainder is 2 when 3^3 is divided by 5, and
- the remainder is 1 when 3^4 is divided by 5.

Calculations involving positive integers n support the conjecture that the remainders when 3^n is divided by 5 repeat in blocks of four as shown above, in which case the desired remainder would have to be 2 since $2019 = 504(4) + 3$.

[C] The key observation is that $n = 4$ is the smallest positive integer for which the remainder is 1 when 3^n is divided by 5.[2] Because $3^4 = 5(16) + 1$ and $2019 = 4(504) + 3$, we have

$$3^{2019} = (3^4)^{504} 3^3 = \big(5(16)+1\big)^{504} 3^3 = \big(5(16)+1\big)^{504} 27.$$

Note that one term in the binomial expansion of $(5(16)+1)^{504}$ is 1 and all the remaining terms are multiples of 5. Consequently, $(5(16)+1)^{504} = 5m + 1$ for some positive integer m, and hence

$$\begin{aligned} 3^{2019} &= (5m+1)27 \\ &= 5(27)m + 27 \\ &= 5(27)m + 5(5) + 2 && \text{Since } 27 = 5(5) + 2 \\ &= 5\big(27m + 5\big) + 2. \end{aligned}$$

Therefore, the desired remainder is 2.

[2] For more information, look up Fermat's little theorem.

9. The format of the choices suggests factoring out $a^{1/2}/a^2$ from each term. For each $a > 0$ with $a \neq 1$, we have

$$\begin{aligned}
\frac{a^{1/2}}{a^2}\left[a^2 - \frac{a^3-1}{a-1} + \frac{a^2+1}{a+1} + 2\right] &= \frac{a^{1/2}}{a^2}\left[a^2 - (a^2+a+1) + \frac{a^2+1}{a+1} + 2\right] \quad \text{Since } a^3-1=(a-1)(a^2+a+1) \\
&= \frac{a^{1/2}}{a^2}\left[1 - a + \frac{a^2+1}{a+1}\right] \\
&= \frac{a^{1/2}}{a^2}\left[\frac{(1-a)(a+1)+a^2+1}{a+1}\right] \\
&= \frac{a^{1/2}}{a^2}\left[\frac{1-a^2+a^2+1}{a+1}\right] \\
&= \frac{2a^{1/2}}{a^2(a+1)}
\end{aligned}$$

and so the answer is **b**.

10. If k is any positive integer, observe that

$$\frac{1}{k(k+1)} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Then

$$\begin{aligned}
\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{8\cdot 9} + \frac{1}{9\cdot 10} &= \sum_{k=1}^{9} \frac{1}{k(k+1)} \\
&= \sum_{k=1}^{9}\left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \cdots + \left(\frac{1}{8}-\frac{1}{9}\right) + \left(\frac{1}{9}-\frac{1}{10}\right) \quad \text{Telescoping sum} \\
&= 1 - \frac{1}{10}
\end{aligned}$$

because all terms except the first and last cancel. Therefore, the answer is 9/10.
