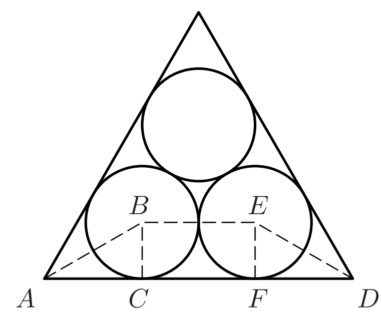
CENTRAL WISCONSIN MATHEMATICS LEAGUE
ANSWER KEYS

Meet II — January 29, 2020

Category I (Geometry, No Calculators)

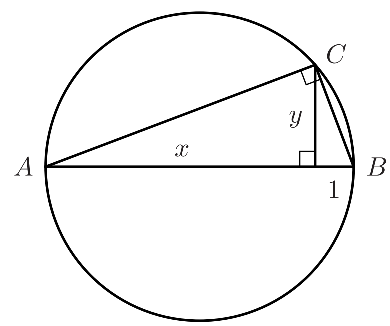
1. **c** 2. **a** 3. **d** 4. **a** 5. **c** 6. **c** 7. **c** 8. **e**

9. Refer to the figure to the right. Observe that $\triangle ABC$ and $\triangle DEF$ are congruent 30°-60°-90° triangles with $BC = EF = 1$ unit. Then $AC = DF = \sqrt{3}$ units and hence the equilateral triangle has side length $s = 2(1+\sqrt{3})$ units since $CF = 2$. It follows that the triangle has area $\sqrt{3}\,s^2/4 = \sqrt{3}(1+\sqrt{3})^2 = \sqrt{3}(4+2\sqrt{3}) = 6+4\sqrt{3}$ square units.



10. Two solutions are given.

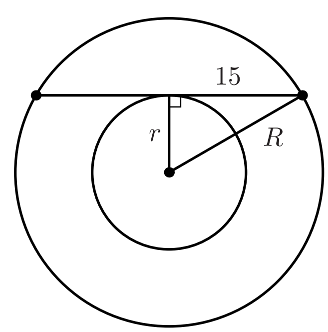
[A] Refer to the figure to the right. Note that $\triangle ABC$ is a right triangle since $\overline{AB}$ is a diameter of the circle. Being the length of an altitude of $\triangle ABC$ that divides the hypotenuse $\overline{AB}$ into segments with lengths x and 1, y is the mean proportional (i.e., the geometric mean) of x and 1 since $y/x = 1/y$ by similar triangles. Therefore, $y = \sqrt{x \cdot 1} = \sqrt{x}$.



[B] Refer to the figure to the right. Let $z = AC$ and $w = BC$. By the Pythagorean theorem, we have $x^2 + y^2 = z^2$, $1^2 + y^2 = w^2$, and $w^2 + z^2 = (x+1)^2$. Solve for y in $(1+y^2) + (x^2+y^2) = (x+1)^2$ to get $y = \sqrt{x}$.

11. Let L denote the initial length of the string in meters, so that $L = 2\pi R$, where R is the radius of the smooth, spherical Earth in meters. If r denotes the radius (in meters) of the circle formed by the shortened string, then $L - 1 = 2\pi r$. Then $L - (L-1) = 2\pi(R - r)$ and therefore $R - r = 1/(2\pi)$ meters.

12. Refer to the figure to the right. Let R and r be the radii (in mm) of the outer and inner circles, respectively. Then $r^2 + (30/2)^2 = R^2$ and so $\pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi(15^2) = 225\pi$ mm^2.



13. The clock's minute hand rotates clockwise $360°/60 = 6°$ each minute, and its hour hand rotates $360°/12 = 30°$ each hour. When the clock reads 10:50, the angle that the minute hand makes with the vertical at 12 is

$(60-50)(6^\circ) = 60^\circ$, and the angle the hour hand makes is $(1+10/60)30^\circ = 35^\circ$. Therefore, the angle between the hands is $60^\circ - 35^\circ = 25^\circ$.

14. Being proportionately-sized, the larger image of Wisconsin has area $(2.5 \text{ cm}/1 \text{ cm})^2 = (5/2)^2 = 25/4$ times the area of the smaller image of Wisconsin. Since the area of the smaller image of Wisconsin is $0.2 = 1/5$ in^2, the area between the two images of Wisconsin is $(25/4)(1/5) - 1/5 = 21/20$ in$^2 \overset{\text{or}}{=} 1.05$ in^2.

Category II (Algebra, No Calculators)

1. **b** 2. **d** 3. **a** 4. **b** 5. **b** 6. **d**

7. Given $n + (n+1) + (n+2) + (n+3) + (n+4) = 2020$, so that $5n + 10 = 2020$, we obtain $n = 402$.

8. The greatest integer a so that $a^6 = (a^3)^2 = (a^2)^3 < 1000$ is $a = 3$. Thus $a^6 = 729$.

9. Since $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot k = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot k = (2^4 \cdot 3^2 \cdot 5)^2 \cdot 7 \cdot k$, the least value is $k = 7$.

10. Any rational roots must be among the values $\pm 1/2$, ± 1, or ± 2. It is easy to check that $x = -1$ is a root, and so we can divide by $x+1$ to see that the polynomial is $(x+1)(2x^3 + x^2 + 3x - 2)$. Likewise, any rational roots of the cubic factor must be among the same values. You can check that a second root is $x = 1/2$ (note that $2(0)^3 + (0)^2 + 3(0) - 2 = -2 < 0$ and $2(1)^3 + (1)^2 + 3(1) - 2 = 4 > 0$ implies there is a real root between 0 and 1). Divide by $2x - 1$ to see that the polynomial is $(x+1)(2x-1)(x^2 + x + 2)$. The quadratic factor clearly has no real roots and therefore the greatest real root of the original polynomial is $x = 1/2$.

11. If x equals the number of correct answers, then $4x - 2(20 - x) = 68$ so that $6x = 108$ and $x = 18$.

12. Rewrite the equation as $(x^2 + 2xy + y^2) - 4x - 4y = 21$ and factor to get $(x+y)^2 - 4(x+y) = 21$. Then $(x+y)^2 - 4(x+y) - 21 = [(x+y)+3]\,[(x+y)-7] = 0$ and thus $x + y = -3$ is the least value.

13. Since $3^{x-1} + 3^x + 3^{x+1} = 351$, so that $3^{x-1}(1+3+9) = 351$, we obtain $3^{x-1} = 27$.

14. Using synthetic division, we obtain

$$\begin{array}{r|rrr} 1 & 1 & b & 3 \\ & & 1 & b+1 \\ \hline & 1 & b+1 & b+4 \end{array} \qquad \text{and} \qquad \begin{array}{r|rrr} -1 & 1 & b & 3 \\ & & -1 & -b+1 \\ \hline & 1 & b-1 & -b+4 \end{array}$$

where it is given that $b + 4 = 2(-b+4)$. Solve for b to get $b = 4/3$.

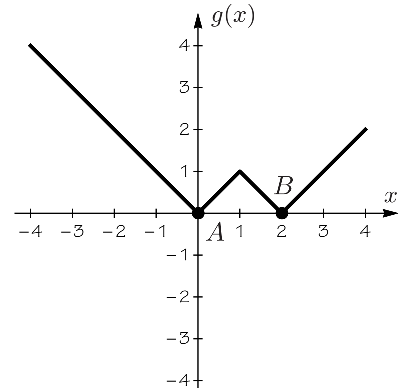
Category III (Advanced Mathematics, No Calculators)

1. Observe that $y - x = 10$.
   (a) **True**
   (b) **True** (Example: $x = 10$, $y = 20$)
   (c) **False** (Example: $x = 11$, $y = 21$)
   (d) **True**
   (e) **False** (See (c))

2. We have $f(1) = -5$ and $f(x+1) = \dfrac{5(x+1)}{(x+1-1)^2 - 1} = \dfrac{5(x+1)}{x^2-1} = \dfrac{5(x+1)}{(x-1)(x+1)} = \dfrac{5}{x-1}$.

   Hence $\dfrac{1}{5}\big[f(x+1) - f(1)\big] = \dfrac{1}{5}\left[\dfrac{5}{x-1} + 5\right] = \dfrac{1}{x-1} + 1 \overset{\text{or}}{=} \dfrac{x}{x-1}$.

3. Since $g(0) = 0 = g(2)$, point A has coordinates $(0, 0)$ and point B has coordinates $(2, 0)$. The graph can be found by first translating the graph of $y = |x|$ one unit to the right and one unit downward to get $y = |x-1| - 1$, and then by reflecting the negative portion across the x-axis to get $y = \big||x-1| - 1\big| = g(x)$. The figure to the right shows the graph when $-4 \le x \le 4$.



   Alternatively, since

$$g(x) = \begin{cases} |1-x-1| = |x| & \text{when } x \le 1, \\ |x-1-1| = |x-2| & \text{when } x \ge 1, \end{cases}$$

   we have

$$g(x) = \begin{cases} -x & \text{when } x \le 0, \\ x & \text{when } 0 \le x \le 1, \\ 2-x & \text{when } 1 \le x \le 2, \\ x-2 & \text{when } x \ge 2. \end{cases}$$

4. Rearrange the inequality to get $n^2(n^2-1) < 11n(n^2-1)$. It follows that $n > 0$ and $n \neq 1$, and so we can divide by $n(n^2-1)$ to get $n < 11$. Since n can be any of 2, 3, 4, 5, 6, 7, 8, 9, or 10, there are 9 solutions for n.

5. In other words, we are counting all three-digit numbers that have digts of 0, 1, 2, and 3 only. Thus the answer is $3 \cdot 4 \cdot 4 = 48$.

6. Observe that $x^3 - 3x^2 + x + 1 = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc$. Hence $a+b+c = 3$ and $abc = -1$. Therefore, $\dfrac{1}{ab} + \dfrac{1}{bc} + \dfrac{1}{ac} = \dfrac{a+b+c}{abc} = -3$.

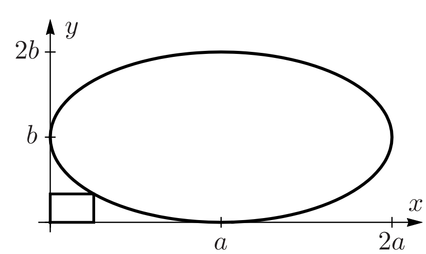
7. If the club has n members of which m are math majors, then the proportion $(n-m)/n$ of non-math majors satisfies $0.91 < (n-m)/n \le (n-2)/n$ since $m \ge 2$. Solve $(n-2)/n > 0.91$ for n to get $n > 22.2$. Therefore, the club has at least 23 members.

8. By rewriting the equation as $(x-3)^2 + y^2 = 10$, we find that the possible integer values for y are ± 3, ± 2, and ± 1. Hence the solution set of ordered pairs (x, y) is $\{(2,-3), (4,-3), (2,3), (4,3), (0,-1), (6,-1), (0,1), (6,1)\}$. Thus the answer is 8.

9. By placing coordinate axes along the walls as shown, the elliptical table corresponds to the equation

$$\frac{(x-a)^2}{a^2} + \frac{(y-b)^2}{b^2} = 1$$

   where $a = 6$ is the semi-major axis length and $b > 1$ is the semi-minor axis length. Letting $x = 6 - \sqrt{20}$ and $y = 1$, and solving the equation for b, we find that $b = 3$. Therefore, the answer is $2b = 6$.



10. Let $n^2 - 65 = m^2$, where n and m are positive integers. Then $n-m$ and $n+m$ are positive integers that satisfy $(n-m)(n+m) = 65$. Hence

$$\begin{cases} n - m = 1 \\ n + m = 65 \end{cases} \quad \text{or} \quad \begin{cases} n - m = 5 \\ n + m = 13 \end{cases}$$

and so by solving the systems of equations, we find that $n = 33$ or $n = 9$.

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